ON MOMENT RELATIONS ON SURFACES OF DISCONTINUITY IN DISSIPATIVE MEDIA

(O MOMENTNYKH SOOTNOSHENIIAKH NA POVERKHNOSTIAKH RAZRYVA V DISSIPATIVNYKH SREDAKH)

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In a number of cases it is known that, within the framework of the model selected to represent the continuous medium, the absence of continuous solutions of the equations of motion requires the introduction of surfaces of discontinuity. On these surfaces, changes in the characteristics of the media and of the motion occur in the form of jumps. In the mechanics of continuous media, surfaces of discontinuity are also used as convenient approximations in narrow zones where the motion of the medium has properties that are essentially different from those in the basic field. At the surface of discontinuity it is necessary, in this and in other cases, to satisfy conditions that allow the continuous solutions on both sides of the surface to be connected. As a rule, these conditions physically mean the specification of a definite quantity of concentrated excitation on the surfaces of discontinuity (forces, sources of matter, energy, etc.) or, in particular, the absence of concentrated excitations on these surfaces. If the surface of discontinuity is approximately represented in a thin region within which the properties of the motion or of the medium differ from those in the basic field, then, generally speaking, in order to determine the magnitude of the concentrated excitations it is necessary to investigate the internal structure of this thin region. Ordinarily, the dynamic conditions on the surfaces of discontinuity are deduced from the laws of conservation of mass, energy, and momentum, taken in integral form. This was first done for arbitrary continuous media in the classical work of Kochin [1].

In many cases of ideal media, the relations for the conservation of mass, energy, and momentum supply the required number of conditions on the surfaces of discontinuity for the specification of the solution.

The situation is different in the case of dissipative media; for such media certain relations for the conservation of mass, energy, and momentum are insufficient. In [2] this circumstance was noted in the case of a viscous fluid. In [2], the conditions of continuity of the tangential component of velocity and temperature were taken as additional conditions on the surface of discontinuity in the boundary layer of a viscous, heat-conducting fluid.

In the present paper, it is shown that the additional relations for dissipative media may be obtained as moment relations of sufficiently high order. In particular, by this means it is shown that one may obtain the conditions of continuity of velocity and temperature in a viscous, heat-conducting fluid. Likewise, it is shown that in the boundary layer surfaces of discontinuity do not exist for the longitudinal components of velocity.

The absence of discontinuities in the tangential components of the velocity is specific to Newtonian viscous fluids; in other dissipative media such discontinuities may exist. In the paper an example is given of a dissipative medium in which an initially present velocity discontinuity does not vanish instantaneously, but rather decays exponentially with time. An analogous situation may take place for temperature discontinuities.

The derivation of additional relations on surfaces of discontinuity has special significance for various models of media with complicated (including higher derivatives) structural dependence of stress on strain, strain rate, etc. The emergence of similar models has increased in recent times with the development of a large number of new materials.

1. We examine the simplest examples at the outset. As is easy to show, the distribution of velocity components u, v and the pressure p in the incompressible viscous fluid (Fig. 1)

$$u = u_1 \quad (y > 0), \qquad u = u_2 \quad (y < 0), \qquad v = 0, \quad p \equiv \text{const} \tag{1.1}$$

satisfies the Navier-Stokes equations outside the surface of discontinuity. Likewise, it satisfies the conditions of conservation of mass, energy, and momentum on the surface of discontinuity $y = 0$.

However, it is easy to see that this distribution is not possible without special external forces, having zero resultant but nonzero resultant moment, applied on the surface of discontinuity. To show this, we note that within an arbitrarily thin transition layer the properties of the fluid are the same as those in the basic flow. Hence, the distribution (1.1) is the limit of a distribution in which $v \equiv 0$ and $p =$ const, as previously, and in which the longitudinal velocity u is a smooth function of y that changes monotonically from u_1 to u_2 on the interval $-h \leqslant y \leqslant h$ and remains constant outside this interval. Here h is an arbitrarily small number. It is clear that any such distribution may be realized in a viscous fluid by the application of external body forces f_x that are directed along the x-axis and distributed according to

$$f_x = -\mu \frac{\partial^2 u}{\partial y^2} \qquad (\mu = \text{const}) \qquad (1.2)$$

(we note that the generalization to the case $\mu \neq$ const poses no difficulties).

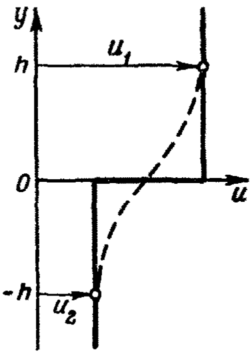

Fig. 1.

From (1.2) and the assumed character of the dependence of u on y it follows that $f_x = 0$ for $|y| > h$. Integrating (1.2) and observing that $\partial u/\partial y = 0$ for $y = \pm h$, $u(h) = u_1$, $u(-h) = u_2$, we find

$$\int_{-h}^{h} f_x \, dy = 0, \qquad \int_{-h}^{h} f_x y \, dy = \mu (u_1 - u_2) \qquad (1.3)$$

Thus, the resultant of the force system applied to the fluid is equal to zero, whereas the resultant moment differs from zero and is independent of the thickness of the intermediate layer $2h$ and of the distribution of the velocity within this layer. Letting h now go to zero, we obtain in the limit the discontinuous distribution (1.1) and verify that an applied external couple of magnitude $\mu(u_1 - u_2)$ per unit area acts on the surface of discontinuity. If the external couple on the surface of discontinuity is absent, then the distribution of longitudinal velocities in the viscous fluid varies continuously upon passage through the surface of discontinuity.

We examine another example, which arises in viscous compressible flow The distribution of the components of velocity, density, and pressure

$$u = u_0 + A \exp(qy / \mu), \quad \rho = \rho_1, \quad v = v_1, \quad p = p_0 - \rho_1 v_1^2 \quad (y > 0)$$
$$u = u_0, \quad \rho = \rho_2, \quad v = v_2, \quad p = p_0 - \rho_2 v_2^2 \quad (y < 0) \qquad (1.4)$$

where p_0, u_0 and A are arbitrary constants and ρ_1, ρ_2, v_1 and v_2 are constants for the auxiliary condition $\rho_1 v_1 = \rho_2 v_2 = q$ (for definiteness we assume $q < 0$), satisfies the equations of motion for viscous compressible flow at $y \neq 0$. This distribution also satisfies the conditions of conservation of mass, momentum, and energy on the surface of discontinuity $y = 0$. By analogy with the above, it is possible to show that an external couple of magnitude μA per unit area acts on the surface of discontinuity. If the external couple is not applied, then at $y = 0$ the longitudinal velocity is continuous ($A = 0$).

However, unlike the previous case, besides the external forces directed along the x-axis and giving rise to the aforementioned couple, there also exist in this example forces f_y, directed along the y-axis in the intermediate layer. To compute the concentrated excitation arising from these forces, we use the momentum equations projected on the y-axis

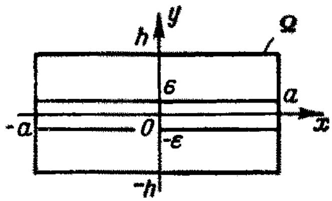

Fig. 2.

$$\frac{\partial \rho v^2}{\partial y} + \frac{\partial p}{\partial y} - \frac{4}{3}\mu\frac{\partial^2 v}{\partial y^2} = f_y \qquad (1.5)$$

Introducing, as before, the smoothed distribution of the velocity v on the interval $-h \leqslant y \leqslant h$, we obtain from (1.5)

$$\int_{-h}^{h} f_y dy = \left(\rho v^2 + p - \frac{4}{3}\mu\frac{\partial v}{\partial y}\right)\Big|_{-h}^{h} = 0 \qquad (1.6)$$

Multiplying equation (1.5) by y and integrating, we obtain likewise

$$\int_{-h}^{h} f_y y dy = (p + \rho v^2)\, y\Big|_{-h}^{h} - \frac{4}{3}\mu\frac{\partial v}{\partial y}y\Big|_{-h}^{h} + \frac{4}{3}\mu(v_1 - v_2) - \int_{-h}^{h}(p + \rho v^2)\, dy \qquad (1.7)$$

Letting h approach zero, we find

$$\int_{-h}^{h} f_y y dy = {}^4/_3\mu\,(v_1 - v_2) \qquad (1.8)$$

Hence, in addition to the couple associated with the discontinuity in the longitudinal velocity, there exists on the surface of discontinuity a concentrated excitation of the "center of pressure" type and of strength $4/3\,\mu(v_1 - v_2)$ per unit area. (The term "center of pressure" denotes the concentrated excitation obtained by applying oppositely-directed normal stresses on two parallel planes and allowing the distance between the planes to go to zero while proportionately increasing the applied stresses.) If such a concentrated excitation is absent from the surface, then the transverse component of the velocity changes continuously.

The examples that have been introduced are obvious proof that surfaces of discontinuity in velocity cannot exist in viscous compressible flow unless the fluid is subjected to concentrated excitations in the form of couples or centers of pressure.

Indeed, the momentum equations projected on the x- and y-axes are of the form

$$\frac{\partial \rho u}{\partial t}+\frac{\partial \rho u^2}{\partial x}+\frac{\partial \rho uv}{\partial y}+\frac{\partial p}{\partial x}-\frac{\mu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f_x$$

$$\frac{\partial \rho v}{\partial t}+\frac{\partial \rho uv}{\partial x}+\frac{\partial \rho v^2}{\partial y}+\frac{\partial p}{\partial y}-\frac{\mu}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=f_y \qquad (1.9)$$

Without loss of generality, the coordinate system can be so chosen that the surface $y = 0$ may be assumed to coincide with the surface of discontinuity within an arbitrarily small neighborhood Ω of the given point O of the discontinuity surface.

We now investigate (Fig. 2) the region $\Omega(-a \leqslant x \leqslant a,\ -h \leqslant y \leqslant h)$. We multiply both parts of equations (1.9) by y and integrate over the region Ω. On the left sides of the integrated equations (1.9), there will occur derivatives of discontinuous functions under the integral sign. To compute these derivatives it is necessary, as was done above, to smooth out the discontinuous functions in a narrow region $-a \leqslant x \leqslant a$, $-\varepsilon \leqslant y \leqslant \varepsilon$, then to compute the integrals in the usual way, and finally to pass to the limit $\varepsilon \to 0$. It can be shown that the result will not depend on the method of smoothing. Estimates show that the results of the integration are of the form

$$\int_\Omega f_x y\,dx\,dy = 2a\mu\,(u_1 - u_2) + O\,(ah) + O\,(a^2)$$

$$\int_\Omega f_y y\,dx\,dy = 2a\,\frac{4}{3}\,\mu\,(v_1 - v_2) + O\,(ah) + O\,(a^2)$$

where the subscript 1 denotes instantaneous values at a point above the surface of discontinuity, while the subscript 2 denotes a point below the surface of discontinuity and $O(c)$ denotes a quantity of order c. We now note that

$$\int_\Omega f_x y dx dy = 2aM + O\,(ah) + O\,(a^2)$$

$$\int_\Omega f_y y dx dy = 2aN + O\,(ah) + O\,(a^2)$$

where M and N are the instantaneous magnitudes of the couple and center of pressure, respectively, on the surface of discontinuity at the point O. Substituting these expressions into the previous equations and passing to the limit $h \to 0$ and $a \to 0$, we obtain

$$M = \mu\,(u_1 - u_2), \qquad N = {}^4/_3\mu\,(v_1 - v_2) \qquad (1.10)$$

In particular, if the surface of discontinuity is free of concentrated excitations, then $M = N = 0$. Hence, we obtain a condition of continuity of the velocities.

2. In the boundary layer the matter is somewhat different from that in viscous flow. Here the coordinates x and y are no longer of equal importance. Besides, they are associated with the body about which the flow takes place, and one is not allowed, in contrast to the previous section, arbitrarily to transform the coordinate system. In a viscous compressible fluid, the equations of the nonstationary boundary layer have the form [3]

$$\frac{\partial\rho u}{\partial t}+\frac{\partial\rho u^2}{\partial x}+\frac{\partial\rho uv}{\partial y}+\frac{\partial p}{\partial x}-\mu\frac{\partial^2 u}{\partial y^2}=f_x$$
$$\frac{\partial p}{\partial y}=0, \qquad \frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}=0 \tag{2.1}$$
$$\frac{\partial\rho cT}{\partial t}+\frac{\partial\rho ucT}{\partial x}+\frac{\partial\rho vcT}{\partial y}-k\frac{\partial^2 T}{\partial y^2}+p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\mu\left(\frac{\partial u}{\partial y}\right)^2=q$$

where c is the specific heat at constant volume, T is the temperature, k is the heat conductivity, and q is the volumetric magnitude of heat generation. The essential fact is that the equation for the v component of velocity markedly simplifies, in contrast to the Navier-Stokes equations, and does not contain the second derivative $\partial^2 v/\partial y^2$. For this reason, discontinuities in the transverse component of velocity v become possible in the boundary layer.

For the derivation of the relations on the surface of discontinuity we shall proceed in analogy to Section 1, starting directly with equations (2.1).

Let us assume that the equation of the surface of discontinuity has the form $y = \Lambda(x, t)$. Without loss of generality, it may be assumed that $\partial\Lambda/\partial t(x_0, t_0) > 0$. We examine the region Ω

$$t_0-\tau\leqslant t\leqslant t_0+\tau, \qquad x_0-a\leqslant x\leqslant x_0+a$$
$$\Lambda(x, t_0-\tau)\leqslant y\leqslant\Lambda(x, t_0+\tau) \tag{2.2}$$

where a and τ are small quantities with respect to an order of smallness that will be specified later, and x_0, t_0 are arbitrary quantities. In the simpler stationary case, the region Ω is specified as follows:

$$x_0-a\leqslant x\leqslant x_0+a, \qquad \Lambda(x_0)-h\leqslant y\leqslant\Lambda(x_0)+h)$$

We note that on both sides of the surface of discontinuity all of the characteristics of the motion are in terms of functions that are continuous with continuous derivatives. By $m_1(x_0, t_0)$ we denote values of certain characteristics of the motion $m(x, y, t)$ at a point $x = x_0$,

$y = \Lambda(x_0, t_0)$, $t = t_0$ immediately above the surface of discontinuity. The value of this quantity directly below the surface of discontinuity we shall denote by $m_2(x_0, t_0)$. Then, in virtue of the smallness of a and τ, the values of m at all points of the region Ω lying below the surface of discontinuity will be close to m_2, and at all points lying above the surface of discontinuity, close to m_1.

We examine next integrals of the form

$$J_1 = \int_\Omega \frac{\partial m}{\partial t}\, dx\, dy\, dt, \qquad J_2 = \int_\Omega \frac{\partial m}{\partial x}\, dx\, dy\, dt$$
$$J_3 = \int_\Omega \frac{\partial m}{\partial y}\, dx\, dy\, dt, \qquad J_4 = \int_\Omega m dx\, dy\, dt \tag{2.3}$$

If the function m undergoes a discontinuity on the surface $y = \Lambda(x, t)$, then the derivatives entering into the first three integrals are generalized functions. Thus, to compute these integrals it is necessary, as was done in Section 1, to "smooth" the function m in a certain small neighborhood of the surface of discontinuity, then to calculate the integral by the usual means, and finally to pass to the limit. In this process it is necessary to take into account the fact that the quantities $\partial\Lambda/\partial x$ and $\partial\Lambda/\partial t$ are of the order $1/\sqrt{\mathrm{Re}}$, and that within the framework of boundary layer theory quantities of this order are considered negligible in comparison to unity. In the limit, the integrals (2.3) turn out to be independent of the manner of smoothing and to be equal to

$$J_1 = 4a\tau \frac{\partial\Lambda}{\partial t}(m_2 - m_1) + O(a^2\tau) + O(a\tau^2)$$
$$J_2 = 4a\tau \frac{\partial\Lambda}{\partial x}(m_2 - m_1) + O(a^2\tau) + O(a\tau^2) \tag{2.4}$$
$$J_3 = -4a\tau(m_2 - m_1) + O(a^2\tau) + O(a\tau^2)$$
$$J_4 = O(a\tau^2)$$

In order to obtain relationships for the conservation of mass, momentum, and energy on the surface of discontinuity, it is sufficient to integrate equations (2.1) over the region Ω, to pass to the limit, $\tau \to 0$ and $a \to 0$, and to take into account the fact that on the surface of discontinuity there are no concentrated loads, sources of mass, or sources of energy. As an example, we carry this out for the law of conservation of momentum. The remaining relations are obtained analogously. Integrating the second of equations (2.1) over the region Ω and using (2.4), we find

$$\int_\Omega \frac{\partial p}{\partial y}\, dx\, dy\, dt = -4a\tau(p_2 - p_1) + O(a^2\tau) + O(a\tau^2) = 0 \tag{2.5}$$

We divide both sides of (2.5) by $4a\tau$ and pass to the limit $\tau \to 0$. In the relation that is obtained we pass to the limit $a \to 0$, and obtain the first relation for the conservation of momentum on the surface of discontinuity [2]

$$p_1 - p_2 = 0 \tag{2.6}$$

Further, integrating the first of equations (2.1) over the region Ω and using (2.4), we obtain

$$4a\tau\left\{\frac{\partial\Lambda}{\partial t}(\rho_2u_2 - \rho_1u_1) + \frac{\partial\Lambda}{\partial x}(\rho_2u_2^2 - \rho_1u_1^2) - (\rho_2u_2v_2 - \rho_1u_1v_1) + \mu\left(\frac{\partial u}{\partial y}\right)_2 - \right.$$
$$\left. - \mu\left(\frac{\partial u}{\partial y}\right)_1 + (p_2 - p_1)\frac{\partial\Lambda}{\partial x}\right\} - \int_\Omega f_x dx\, dy\, dt + O(a\tau^2) + O(a^2\tau) = 0 \tag{2.7}$$

We now note that

$$\int_\Omega f_x dx\, dy\, dt = 4a\tau F(x_0, t_0) + O(a\tau^2) + O(a^2\tau) \tag{2.8}$$

where F is the magnitude of the concentrated load on the surface of discontinuity, and, by virtue of (2.6), $p_1 - p_2 = 0$. In addition, it is obvious that $\partial\Lambda/\partial t = D$, $\partial\Lambda/\partial x = \tan\beta$, where D is the velocity of propagation of the surface of discontinuity and β is the angle of its inclination with the x-axis. Substituting these relations into (2.7) and, in analogy to the foregoing, passing to the limit $\tau \to 0$ and $a \to 0$, we obtain the second relation for the conservation of momentum on the surface of discontinuity in the form

$$\rho_1(D + u_1\tan\beta - v_1)u_1 + \mu\left(\frac{\partial u}{\partial y}\right)_1 + F =$$
$$= \rho_2(D + u_2\tan\beta - v_2)u_2 + \mu\left(\frac{\partial u}{\partial y}\right)_2 \tag{2.9}$$

If there are no concentrated forces, then $F \equiv 0$, and the relation (2.9) takes on the form given in [2].

Similarly, by integrating the third and fourth of equations (2.1) we obtain relations for the conservation of mass and energy.

As was mentioned at the outset, a single one of the relations for the conservation of mass, energy, or momentum is insufficient for the unique specification of the solution on both sides of the surface of discontinuity. The additional relation is obtained from the examination of still another conservation law – the law of conservation of angular momentum. To obtain the relation for the conservation of angular momentum, we multiply both sides of the first of equations (2.1) by $y - \Lambda(x_0, t_0)$ and integrate over the region Ω. Transforming the y derivatives by means of the formulas

$$[y - \Lambda(x_0, t_0)]\frac{\partial \rho uv}{\partial y} = \frac{\partial [y - \Lambda(x_0, t_0)]\,\rho uv}{\partial y} - \rho uv$$
$$[y - \Lambda(x_0, t_0)]\,\mu\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left\{[y - \Lambda(x_0, t_0)]\mu\frac{\partial u}{\partial y}\right\} - \mu\frac{\partial u}{\partial y}$$

noting that the functions

$$\rho u\,[y - \Lambda(x_0, t_0)],\ \rho u^2\,[y - \Lambda(x_0, t_0)],\ \rho uv\,[y - \Lambda(x_0, t_0)]$$
$$p\,[y - \Lambda(x_0, t_0)],\ \frac{\partial u}{\partial y}[y - \Lambda(x_0, t_0)]$$

vanish on the surface of discontinuity itself and are continuous in its neighborhood, and using relations (2.4), we find

$$4a\tau\mu\,(u_1 - u_2) + O(a^2\tau) + O(a\tau^2) - \int_\Omega f_x\,[y - \Lambda(x_0, t_0)]\,dx\,dy\,dt = 0 \quad (2.10)$$

But, in analogy with the foregoing

$$\int_\Omega f_x[y - \Lambda(x_0, t_0)]\,dx\,dy\,dt = 4a\tau M + O(a^2\tau) + O(a\tau^2) \quad (2.11)$$

where M is the magnitude of the moment of the concentrated couple on the surface of discontinuity. Substituting this into (2.10) and passing to the limit $\tau \to 0$ and $a \to 0$, we obtain

$$\mu\,(u_1 - u_2) = M \quad (2.12)$$

so that the jump in the longitudinal component of velocity is proportional to the magnitude of the couple applied on the surface of discontinuity. In particular, if the couple is absent (free surface of discontinuity), then the longitudinal component of velocity must be continuous.

The condition of continuity of the temperature $T_1 = T_2$ is obtained by an almost literal repetition, as applied to the last of equations of the system (2.1), of the arguments used in the derivation of the conditions of continuity of the tangential velocity. Instead of the condition of the absence of an external couple on the surface of discontinuity, we use here the condition of the absence of so-called concentrated "heat dipoles" on the surface of discontinuity. The term "heat dipole" is understood to mean, as is usual, the concentrated excitation which is obtained in the limit by placing heat sources of opposite signs at a certain distance apart, and then decreasing to zero the distance between the sources while proportionately increasing the sources' strength. The absence of temperature dipoles may also be treated as the absence of concentrated thermal resistance on the surface of discontinuity.

3. The arguments that have been introduced can be extended directly to an extremely wide class of dissipative media, for which the relation between the stress and strain rate tensors can be written in the form

$$\tau_{ij} = f\left(\varepsilon_{kl}, A_\alpha \frac{\partial \varepsilon_{kl}}{\partial x_\alpha}, \; A_{\alpha\beta} \frac{\partial^2 \varepsilon_{kl}}{\partial x_\alpha \partial x_\beta}, \; \ldots, \; \dot{\varepsilon}_{kl}, \; B_\alpha \frac{\partial \dot{\varepsilon}_{kl}}{\partial x_\alpha}, \ldots\right) \tag{3.1}$$

where f is an arbitrary, sufficiently smooth function which admits of the required invariance. In such media, the conditions of conservation of mass, energy, and momentum on the surface of discontinuity are, generally speaking, insufficient, and it is necessary to obtain additional conditions from moment relations of higher order - analogously to what was done above for a viscous fluid.

As an example, we consider a medium for which the relation between the stress and the strain rates is of the form

$$\tau_{ij} = -p\delta_{ij} + \mu\varepsilon_{ij} + A_{\alpha\beta} \frac{\partial^2 \varepsilon_{ij}}{\partial x_\alpha \partial x_\beta} \tag{3.2}$$

In the one-dimensional case ($u = u(y, t)$, $v \equiv 0$) we have

$$\tau_{xy} = \mu \frac{\partial u}{\partial y} + \varkappa \frac{\partial^3 u}{\partial y^3} \tag{3.3}$$

so that the basic dynamical equation is of the form

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \varkappa \frac{\partial^4 u}{\partial y^4} \tag{3.4}$$

As previously, it may be shown that to determine the additional conditions on the surface of discontinuity it is necessary to bring in moment relations not only of zero and first order, but also of second and third order. In this case it is necessary to use the condition that the surface of discontinuity is free of concentrated excitations, or, to specify these excitations from an examination of the structure of a thin test region of the surface of discontinuity. For a discontinuity surface free of concentrated excitations, the conditions of continuity of velocity and its first derivative are obtained from moment relations of second and third order.

It should be stressed that the absence of discontinuities in velocity, in particular the instantaneous vanishing of discontinuities excited by any means, is specific to Newtonian viscous fluids. In other dissipative media, excited discontinuities do not, generally speaking, disappear, so that discontinuities may also exist in the absence of concentrated excitations on the surface of discontinuity. For example, consider a medium with an equation of state of the form

$$\tau_{ij} = -p\delta_{ij} + \mu\varepsilon_{ij} + \eta\dot{\varepsilon}_{ij} \tag{3.5}$$

In the one-dimensional case we have

$$\tau = \mu \frac{\partial u}{\partial y} + \eta \frac{\partial^2 u}{\partial y\, \partial t} \tag{3.6}$$

so that the basic dynamical equation takes on the form

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \eta \frac{\partial^3 u}{\partial y^2 \partial t} \tag{3.7}$$

Repeating the arguments, introduced in the previous sections, to this equation (see also [4]), we find that the following relations are satisfied on a surface of discontinuity free of concentrated excitations:

$$\mu u_1 + \eta \frac{\partial u_1}{\partial t} = \mu u_2 + \eta \frac{\partial u_2}{\partial t}, \qquad \mu \left(\frac{\partial u}{\partial y}\right)_1 + \eta \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)_1 = \mu \left(\frac{\partial u}{\partial y}\right)_2 + \eta \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)_2 \tag{3.8}$$

where, as before, the subscripts 1 and 2 denote values of quantities on different sides of the discontinuity surface. Integrating (3.8), we obtain

$$u_1 - u_2 = (u_1 - u_2)_{t=t_0} e^{-\mu (t-t_0)/\eta}, \quad \left(\frac{\partial u}{\partial y}\right)_1 - \left(\frac{\partial u}{\partial y}\right)_2 = \left[\left(\frac{\partial u}{\partial y}\right)_1 - \left(\frac{\partial u}{\partial y}\right)_2\right]_{t=t_0} e^{-\mu (t-t_0)/\eta} \tag{3.9}$$

so that, in such a medium, jumps that are excited for any reason whatever decay exponentially with time for $\eta \neq 0$, and do not disappear instantaneously as they do in a Newtonian viscous fluid ($\eta = 0$).

It may be remarked that in all cases the identical method of "smoothing" was used to derive the concentrated discontinuities on the surfaces of discontinuity. The introduction of external excitations in the region of smoothing means, mathematically, that in addition to the homogeneous equations, the corresponding equations with right-hand sides are considered. As a result, the passage to the limit as the thickness of the region of smoothing goes to zero determines a limiting form of the relation between jumps in the characteristics of the motion on the surface of discontinuity and integral characteristics of the right-hand sides of the equations - the concentrated excitations. In a number of cases, the application of the theory of generalized functions, which has been well developed in recent times [5, 6], avoids the necessity of introducing intermediate arguments. A fundamental role in the theory of generalized functions is played by the delta function, which, from a physical point of view, represents a concentrated excitation (whose type is that of a concentrated force, concentrated flow of energy, mass, etc.). Successive differentiations of the delta function lead to concentrated excitations of other types (moments, temperature dipoles, etc.).

It can be shown [6, p.149] that all concentrated excitations may be represented as a linear combination of delta functions and its derivatives in the right-hand side of the respective equations, so that these equations take on the form

$$R(u_1,\ u_2 \ldots) = A\delta + B\delta' + C\delta'' + \ldots \qquad (3.10)$$

where u_1, u_2, ... are unknown functions, and R is a differential operator corresponding to the equation being considered. Since the order of this operator is bounded and all of the unknown functions are piecewise continuous, having at most discontinuities of the first type, the number of terms in the linear combination of the delta functions and its derivatives on the right-hand side of (3.10) is finite. The coefficients A, B, C, ... are quantities that are applied to the system of concentrated excitations and, in accordance with the definition of the delta function and its derivatives, may be found by means of moment relations as was done above.

The number of moment relations necessary for the unique specification of discontinuous solutions is equal to the order of the highest derivative of the delta function on the right-hand side of (3.10).

4. As has already been mentioned above, the surface of discontinuity may be studied by means of a thin test region, where the properties of the motion or of the medium change abruptly. In such cases concentrated excitations of one form or another may be generated not only by external means but by internal motions within the thin test region itself. Here, in order to obtain all of the relations on the surface of discontinuity, among them the concentration of mass, energy, and momentum, it is necessary to investigate the internal structure of the test region associated with the surface of discontinuity, or, as has been done in many cases, to introduce additional hypotheses as to the quantity of concentrated excitation entering into the equations of mass, momentum, and energy and into the moment relations.

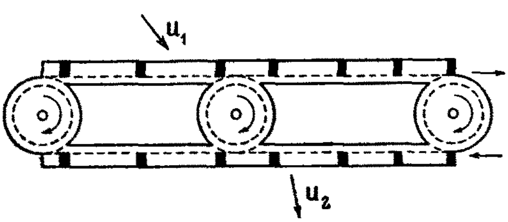

Fig. 3.

We consider some examples. In the introduction of jumps of condensation and detonation in an ideal gas, the heat generation is assumed to be given in the relationships for the conservation of energy on the surface of discontinuity. In reality, this is specified by processes with complicated kinetics that take place in a narrow zone.

The impossibility of discontinuities in the normal component of velocity in a viscous fluid in the absence of concentrated excitations does not contradict, for example, the possibility of jumps of condensation and change in density in a compressible, viscous fluid. The process of condensation brings about a change in the equations of motion (at the expense of a change in the dependence of the stress tensor on strain rate) which in the limit is equivalent to the introduction of a concentrated excitation in a viscous fluid. The difficulty in studying the internal structure of the region of condensation has to do with the representation, for example, of the jump in density (which is equivalent to the specification of a concentrated excitation of the center of pressure type). In hydraulic investigations of so-called local resistances in pipes (gates, nets, diaphragms, etc.), these resistances are replaced by surfaces of discontinuity. In the relations for the conservation of momentum on such a surface of discontinuity, the magnitude of the concentrated force acting on the moving fluid is assumed given. Finally, we consider the motion of a viscous fluid through two closely-spaced parallel permeable plates (Fig. 3). The plates move, without sliding, one with respect to the other, by means of rolls to which couples are applied. Replacing the system of plates and rolls by a surface of discontinuity, it is necessary in the relation for the conservation of momentum to specify the moment of the concentrated couples on the surface of discontinuity.

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